A modulus and an extremal form of a foliation

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We define a $p$-modulus of a subset of leaves of a foliation on Riemannian manifold. We prove, that the $p$-modulus of the foliation is conformal invariant and study the problem of existing of the extremal form for a foliation on Riemannian manifold. We also compute the value of $p$-modulus and the extremal form for $k$-dimensional foliation given by a submersion.

The idea of modulus is directly connected with the concept of extremal length of curves in $\mathbb{R}^2$ introduced by Beurling and Ahlfors [AhBe] in the beginning of 50-ties. In 1957 Fuglede generalized this notion to the modulus of $k$-dimensional surface families in $\mathbb{R}^n$. It was very useful tool in the theory of conformal and quasiconformal maps, extremely popular in 60-ties and 70-ties.

Using a geometric characterization Suominen [Su] extended the modulus to the case of an arbitrary differential Riemannian manifold, and in 1979 Krivov [Kr] defined generalized $p$-modulus for a family of $k$-forms.

The modulus of a foliation, introduced by the author in [Bl], connects Fuglede’s and Krivov’s ideas. We used the fact that the foliation of Riemannian manifold may be defined as a family of surfaces or by a family of forms. The modulus of the foliation is just a modulus in Krivov’s sense of the family of forms characteristic for the foliation. For these forms, by Hodge star, arises the family of dual forms. Both these classes seem to characterize pairs of foliations orthogonal to each other, but this is an open problem yet.

Let $(M, \mathcal{F})$ be a smooth oriented foliated Riemannian $n$-manifold and $\dim \mathcal{F} = k$. We denote by $\mathcal{L}_p^k(M)$ ($p \geq 1$) the space of measureable and $p$-integrable $k$-forms $\omega$ on $M$ with norm $\|\omega\| = \left(\int_M |\omega(x)|^p \sigma_M \right)^{\frac{1}{p}}$, where $\sigma_M$ is the volume form of $M$.

Let $\mathcal{L} \subset \mathcal{F}$. By $\operatorname{adm}(\mathcal{L})$ we denote the family of all $k$-forms on $M$ such that $\omega \in \mathcal{L}_p^k(M)$, $\int_L \omega \geq 1$ for almost every leaf $L \in \mathcal{L}$ and $\omega$ is almost everywhere positively definable (i.e. for almost every $x \in M$ and for every orthonormal positive oriented base $e_1, \ldots, e_k \in T_x \mathcal{F}$ $\omega(e_1, \ldots, e_k)$ is positive). Elements of $\operatorname{adm}(\mathcal{L})$ we call admissible forms for $\mathcal{L}$. 

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The \( p \)-modulus of \( \mathcal{L} \) is defined as \( \operatorname{mod}_p(M, \mathcal{L}) = \inf_{\omega \in \text{adm}(\mathcal{L})} \| \omega \| \).
If there exists an admissible form \( \omega \) such that \( \| \omega \| = \operatorname{mod}_p(M, \mathcal{L}) \) we call it an extremal form for \( \mathcal{L} \) and denote by \( \omega_0(\mathcal{L}) \). If \( \mathcal{L} = \mathcal{F} \) we have a modulus of a foliation \( \mathcal{F} \).

The modulus has some useful properties:

1. it is monotone and countable subadditive, i.e.
   \[
   \operatorname{mod}_p(M, \mathcal{L}_1) \leq \operatorname{mod}_p(M, \mathcal{L}_2), \quad \text{if} \quad \mathcal{L}_1 \subset \mathcal{L}_2,
   \]
   \[
   \left( \operatorname{mod}_p(M, \mathcal{L}) \right)^p \leq \sum_{i \in \mathbb{N}} \left( \operatorname{mod}_p(M, \mathcal{L}_i) \right)^p, \quad \text{if} \quad \mathcal{L} = \bigcup_{i \in \mathbb{N}} \mathcal{L}_i,
   \]

2. if \( N \) is an open subset of \( M \) and \( \mathcal{L} \subset \mathcal{F} \), then
   \[
   \operatorname{mod}_p(M, \mathcal{L} \cap \tilde{N}) \leq \operatorname{mod}_p(N, \mathcal{L}|N),
   \]
   where \( \mathcal{L}|N = \{L \cap N, L \in \mathcal{L}\} \) and \( \tilde{N} \) is the saturation of \( N \) in \( \mathcal{F} \),

3. if \( \mathcal{L} \subset \mathcal{F} \) and \( N_1, N_2 \) are open subsets of \( M \), such that for almost every leaf \( L \in \mathcal{L} \) hold \( L \cap N_1 \neq \emptyset \) and \( L \cap N_2 \neq \emptyset \), then
   \[
   \left( \operatorname{mod}_p(M, \mathcal{L}) \right)^{\frac{p}{p-1}} \geq \left( \operatorname{mod}_p(N_1, \mathcal{L}|N_1) \right)^{\frac{p}{p-1}} + \left( \operatorname{mod}_p(N_2, \mathcal{L}|N_2) \right)^{\frac{p}{p-1}},
   \]

4. A family \( \mathcal{L} \subset \mathcal{F} \) is \( p \)-exceptional if and only if there exists an admissible form \( \omega \) for \( \mathcal{L} \) such, that \( \int_L \omega = \infty \) for almost all leaves \( L \in \mathcal{L} \),

5. If the volume of \( M \) is finite, then the family of leaves
   \( \mathcal{L} = \{L \in \mathcal{F}; \text{vol} L = \infty\} \) is \( p \)-exceptional.

6. For \( p = n/k \) \( p \)-modulus of foliation is a conformal invariant.

Now we will say something about an extremal form and value of modulus of foliation in a special case.

**Theorem.** If foliation \( \mathcal{F} \) on \( M \) is given by a submersion \( f \) with connected level sets, then

\[
\operatorname{mod}_p^p(M, \mathcal{F}) = \int_{\mathcal{F}(M)} \left( \int_{L_x} J_f^{\frac{1}{p-1}} \sigma_{L_x} \right)^{1-p} \sigma_f(M),
\]

and \( k \)-form on \( M \)

\[
\omega_0(x) = \frac{J_f^{\frac{1}{p-1}}}{\int_{L_x} J_f^{\frac{1}{p-1}} \sigma_{L_x}} \sigma_{L_x}
\]

is an \( p \)-extremal form for \( \mathcal{F} \).
As a simple consequence of above theorem we receive, that if $F$ is codimen-
sion one foliation given by a submersion $f : M \to (a,b) \subset \mathbb{R}$ with connected
level sets, then

$$\text{mod}_p^p(M,F) = \int_a^b \left( \int_{L_x} \|\text{grad} f\|^{\frac{1}{p-1}} \sigma_{L_x} \right)^{1-p} dt ,$$

where $f(x) = t$, and $(n-1)$-form on $M$

$$\omega_0(x) = \frac{\|\text{grad} f\|^{\frac{1}{p-1}}}{\int_{L_x} \|\text{grad} f\|^{\frac{1}{p-1}} \sigma_{L_x}} \sigma_{L_x}$$

is an $p$-extremal form for foliation $F$.

In last two theorems we describe properties of an extremal form.

**Theorem.** If $\omega_0$ is an extremal form of a family $\mathcal{L} \subset F$, then for almost
all leaves $L \in \mathcal{L}$ holds

$$\int_L \omega_0 = 1 .$$

**Theorem.** If an extremal form $\omega_0$ of a foliation $F$ is continous, then for
all vector fields $X_1, \ldots, X_{n-k} \in TF$

$$\ast \omega_0(X_1, \ldots, X_{n-k}) = 0 .$$

That means in particular, that for each leaf $L \in F$ there exists a real function $f_L$ on $M$ such that $\omega_0|_L = f_L \cdot \sigma_L$, where $\sigma_L$ is the volume form of $L$.

**References**

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