REALIZATIONS OF GROWTH TYPES

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1. INTRODUCTION

If we consider the set of all growth types then it is natural to ask when a given growth type can be realized as a growth type of some leaves of foliated manifolds. Growth of leaves plays an important role in the study of topology and dynamics of foliations.

An interesting problem is how to obtain leaves of a given growth type on some compact foliated manifold. The set of growth types contains many types which cannot be compared with polynomial, fractional or exponential ones. Many examples of leaves with growth types between polynomial ones have been described by J. Cantwell and L. Conlon in [CC1], [CC2], [CC3] and by Tsuchiya in [Ts]. Our family of growth types realizable by leaves seems to be essentially larger than those considered in mentioned papers but we had to pay some price. Our foliations are only $C^1$-differentiable and of codimension 2.

We will show that any growth type not greater than the exponential and satisfying simple conditions (described in [Ba]) can be realized as a growth type of an orbit of finitely generated group of diffeomorphisms on the torus and as a growth type of leaves of a compact foliated manifold.

2. GROWTH TYPES

In this section recall the notion of the growth type (compare [HH], [CdC]) and some results from [Ba].

Let $\mathcal{I}$ be the set of nonnegative nondecreasing functions on $\mathbb{N}$:

$$\mathcal{I} = \{ f : \mathbb{N} \rightarrow \mathbb{R}_+ : f(n) \leq f(n+1) \text{ for all } n \in \mathbb{N} \}.$$ 

Define a preorder $\preceq$ in $\mathcal{I}$. Let $f, h \in \mathcal{I}$. We say that $h$ dominates $f$ (we write $f \preceq h$) if and only if for some $A \in \mathbb{R}_+$ and $B \in \mathbb{N}$

$$f(n) \leq Ah(Bn) \quad \text{for any } n \in \mathbb{N}.$$ 

The preorder $\preceq$ induces an equivalence relation $\simeq$ in $\mathcal{I}$:

$$f \simeq h \iff f \preceq h \preceq f$$

The equivalence class of $f \in \mathcal{I}$ is called the growth type of $f$ and is denoted by $[f]$. We will denote by $\mathcal{E}$ the set of all equivalence classes in $\mathcal{I}$. $\mathcal{E}$ has the partial order $\leq$ induced by the preorder $\preceq$. If $f \preceq h$, then $[f] \leq [h]$ and if $f \preceq h \preceq f$, then $f$ and $h$ have the same growth type and $[f] = [h]$. If $f \preceq h$, but $h \not\preceq f$, then we write $[f] < [h]$. 

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For example, we can easily see that
\[ [0] < [1] = [2] < [n] < [n^2] < \cdots < [2^n] = [3^n] < [2^{2n}] . \]
For \( k \geq 0, [n^k] \in \mathcal{E} \) is called the polynomial growth of degree \( k \). For any \( a > 1, [a^n] \) is equal to \([e^n]\) and is called exponential growth. We will denote exponential growth by \([\exp]\).

For two functions \( f, h \in \mathcal{I} \) defined by
\[
 f(n) = n, \\
 h(n) = \begin{cases} 
 (k + 2)^{k+2} & \text{if } k^k \leq n < (k + 4)^{k+4} \text{ and } k = 1, 5, 9, \ldots 
\end{cases}
\]
the growth types \([f]\) and \([h]\) are incomparable.

Lemma 2.1. ([Ba]) For any growth types \( \xi, \eta \in \mathcal{E} \) such that \([0] < \xi < \eta\) there exists a growth type \( \vartheta \in \mathcal{E} \) such that \( \xi < \vartheta < \eta\).

Let us recall from [Ba] definitions of nice and derived growth types. Let \( \xi, \eta \in \mathcal{E} \). We say that \( \eta \) is the derived growth type of \( \xi \) if
\[
 [\Sigma f] = \xi \quad \text{for any } f \in \eta,
\]
where \( \Sigma f(n) = \sum_{k=1}^{n} f(k) \).

For example,
\[
 [\Sigma 1] = [n], \quad [\Sigma n^k] = [n^{k+1}], \\
 [\Sigma \exp] = [\exp], \quad [\Sigma n^{\alpha}] = [n^{\alpha+1}], \text{ where } \alpha \in \mathbb{R}_+.
\]

Lemma 2.2. ([Ba]) Let \( f, h, F, H \in \mathcal{I} \) and \( F = \Sigma f, \quad H = \Sigma h \). Then
- \([F] = [H]\) if and only if \([f] = [h]\),
- \([F] < [H]\) if and only if \([f] < [h]\),
- \([F], [H]\) are incomparable if and only if \([f], [h]\) are.

We say that a growth type \( \xi \in \mathcal{E} \) is nice if there exist \( f \in \xi \) and a positive integer \( p \) such that
\[
 f(n + 1) \leq pf(n) \quad \text{for all } n \in \mathbb{N}.
\]

Lemma 2.3. ([Ba]) If \( \xi \leq [\exp] \), then \( \xi \) is nice.

Lemma 2.4. For a nice growth type \( \xi > [0] \) there exist integer-valued \( h \in \xi \) and \( q \in \mathbb{N} \) such that for each \( n \in \mathbb{N} \) there exists \( q_n \in \{1, 2, \ldots, q\} \) such that
\[
 h(n + 1) = q_n h(n).
\]

Now we recall the notions of growths of Riemannian manifolds, leaves and orbits of group actions (pseudogroups).

Let \((M, g)\) be a complete connected Riemannian manifold. Fix a point \( x \in M \) and define
\[
 f_x : \mathbb{N} \to \mathbb{R}_+,
\]
the growth function of \( M \) at \( x \), by
\[
 f_x(n) = \text{Vol}(B(x,n)),
\]
where \( B(x,n) \) is the centered at \( x \) ball of radius \( n \) on \( M \) and \( \text{Vol} \) is the measure (volume) on \( M \) induced by the Riemannian structure \( g \).
Obviously $f_x$ belongs to $I$. If $y$ is another point of $M$, then $B(y, n) \subset B(x, n + l)$ and $B(x, n) \subset B(y, n + l)$, where $l \geq \text{dist}(x, y)$ and dist is the distance function on $(M, g)$. Therefore, if $f_y$ is the growth function of $M$ at $y$, then

$$f_x(n) \leq f_y((l + 1)n) \quad \text{and} \quad f_y(n) \leq f_x((l + 1)n) \quad \text{for all } n \in \mathbb{N}.$$ 

Hence $[f_x] = [f_y]$. The equivalence class $[f_x]$ is called the growth type of $(M, g)$ and is denoted by $\text{gr}(M)$.

Let $\mathcal{F}$ be a foliation of a Riemannian manifold $M$. A leaf $L \in \mathcal{F}$ inherits a Riemannian metric from $M$. So, one can define the growth type of the leaf analogously as the growth type of Riemannian manifolds.

Now let $G$ be a finitely generated group and let $G_1 = \{g_1, \ldots, g_n\}$ be a finite, symmetric generating set containing the identity $e$. Each $g \in G$ has a representation of the form

$$g = g_{i_1}g_{i_2}\cdots g_{i_n}$$

We define the length $|g|$ of $g \in G$ to be the smallest integer $n \geq 1$ for which such a representation exists. Let $G_n = \{g \in G : |g| \leq n\}$ for each $n \geq 1$.

Suppose that $G \times X \to X$ is a group action of $G$ on the set $X$. If $x \in X$, we can define $f_x : \mathbb{N} \to \mathbb{R}_+$, the growth function of the orbit $G(x)$ by

$$f_x(n) = \#G_n(x) = \#\{g(x) : g \in G_n\}.$$ 

The growth type of the growth function of the orbit $G(x)$ does not depend on the choice of finite, symmetric generating set of $G$ nor on the choice of $y \in G(x)$ (see [CdC]). The equivalence class $[f_x]$ is called the growth type of the orbit and is denoted by $\text{gr}(G(x))$.

Similarly, if $G$ is a finitely generated pseudogroup of homeomorphisms on a topological space $X$ and $G_1$ is a finite symmetric generating set containing the identity $\text{id}_X$, then, if $x, y \in X$ lie in a common $G$ - orbit, we can write

$$y = h_1 \circ h_2 \circ \cdots \circ h_n(x),$$

where all $h_i \in G_1$. Define the distance $d(x, y)$ to be the smallest integer $n$ for which such an expression exists. The growth function $f_x : \mathbb{N} \to \mathbb{R}_+$ of the orbit $G(x)$ is defined by

$$f_x(n) = \#\{y \in G(x) : d(x, y) \leq n\}.$$ 

Suppose that $\mathcal{U}$ is any regular covering of a compact foliated manifold $(M, \mathcal{F})$. Let $\mathcal{H}$ be the holonomy pseudogroup corresponding to the covering $\mathcal{U}$. $\mathcal{H}$ is generated by a finite symmetric set $\mathcal{H}_1$ and the growth type of the growth function $f_x$ of the orbit $\mathcal{H}(x)$ does not depend on the choice of $x \in \mathcal{H}(x)$ (see [CdC], [Wa]). The equivalence class $[f_x]$ is called the growth type of the holonomy orbit and is denoted by $\text{gr}(\mathcal{H}(x))$. 
Lemma 2.5. If \( H \) and \( H' \) are the holonomy pseudogroups corresponding to two regular coverings \( \mathcal{U} \) and \( \mathcal{U}' \) of a compact foliated manifold \((M, \mathcal{F})\), then \( \text{gr}(H(x)) = \text{gr}(H'(x)) \) for any point \( x \in M \).
(See \[Pl\], \[CdC\], \[Wa\]).

If \( \mathcal{F} \) is a foliation of compact manifold \( M \) then the growth of a leaf \( L \) of \( \mathcal{F} \) is equal to the growth of the corresponding holonomy orbit.

Lemma 2.6. Let \( H \) be the holonomy pseudogroup of a compact foliated manifold \((M, \mathcal{F})\). Then \( \text{gr}(H(x)) = \text{gr}(L) \), where \( L \) is the leaf of \( \mathcal{F} \) passing through \( x \) equipped with any Riemannian metric inherited from \( M \).
(See, for example \[Pl\], \[CdC\], \[Wa\]).

So, for leaves of compact foliated manifolds growth also can be defined, without a metric, in terms of the growth at \( x \in L \) of the holonomy pseudogroup of \( \mathcal{F} \).

3. Realization of growth types

Our goal in this section is to realize a growth type by orbits and leaves. We construct a group of diffeomorphisms (of the torus) for which the growth type of a suitable orbit is equal to a given growth type (Theorem 3.4). Finally, in Corollary 3.5 we show that if a growth type (not greater than exponential one) has a derived one then it can be realized as a growth type of a leaf. We construct (by suspension) a compact foliated manifold with a leaf of a given growth type.

The following theorem was proved in \[Ba\].

Theorem 3.1. ([Ba]) If a growth type \( \xi \leq \exp \) has a derived growth type \( \eta \), then there exists a complete connected Riemannian manifold \((M, g)\) with bounded geometry, such that \( \text{gr}(M) = \xi \).

Now we describe two examples of diffeomorphisms, the first one the plane \( \mathbb{R}^2 \) and the second one of the torus \( T^2 \). We will use these examples for a construction of a finitely generated group of diffeomorphisms of the torus \( T^2 \).

Example 3.2. Take \( r > 0 \) and a \( C^\infty \)-function \( u: \mathbb{R} \rightarrow \mathbb{R} \) such that
(i) \( u(t) = 0 \) for \( t \leq r \) or \( t \geq 3r \),
(ii) \( u(2r) = 1 \) and \( u(t) < 1 \) for \( t \neq 2r \),
(iii) \( 0 \leq u'(t) \leq \frac{A}{r} \) for \( t \leq 2r \)
\[-\frac{A}{r} \leq u'(t) \leq 0 \] for \( t \geq 2r \),
where \( A \in \mathbb{R} \) is constant.
Now, consider \( g: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by
\[
g(x, y) = (g_1(x, y), g_2(x, y)) = (x \cos(u(\sqrt{x^2 + y^2}) \beta) - y \sin(u(\sqrt{x^2 + y^2}) \beta), x \sin(u(\sqrt{x^2 + y^2}) \beta) + y \cos(u(\sqrt{x^2 + y^2}) \beta)),
\]
where $\beta \in [0, 2\pi)$ is a constant. Note that,

- $g(x, y) = (x, y)$, when $\sqrt{x^2 + y^2} \leq r$ or $\sqrt{x^2 + y^2} \geq 3r$,
- $gC_{2r}$ is the rotation of angle $\beta$, where $C_{2r}$ is the circle centered at $(0, 0)$ of radius $2r$.

One can see that $g$ is a diffeomorphism of the plane and

$$\lim_{\beta \to 0} g'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first part of the next example consists in a suitable modification of the Denjoy diffeomorphism of the circle (see for example [Ta]).

**Example 3.3.** Let $I = [0, 1]$ and $\alpha$ be an irrational number. Consider a set $\{I_m = [0, l_m] : m \in \mathbb{Z}, l_m > 0\}$ such that

(i) $\sum_{m \in \mathbb{Z}} l_m = l < \infty$,
(ii) $\lim_{m \to \pm \infty} \frac{l_{m+1}}{l_m} = 1$.

We can set, for example, $l_m = \frac{1}{1 + m^2}$. For each $m \in \mathbb{Z}$, let $\alpha_m = \{ma\}$ be the fractional part of $ma$, i.e. $0 \leq \alpha_m < 1$ and $ma - \alpha_m \in \mathbb{Z}$. Moreover $\alpha_n \neq \alpha_m$ when $n \neq m$.

For each $m \in \mathbb{Z}$ we open $I$ at the point $\alpha_m$ and glue in there the segment $I_m$. We obtain a segment $J$ of length $1 + l$,

$$J = I \cup \bigcup_{m \in \mathbb{Z}} I_m.$$

Next, choose differentiable functions $f_m$ mapping $I_m$ onto $I_{m+1}$ in such a way that

(i) $f'_m(x) > 0$,
(ii) $f'_m(x) = 1$ in some neighbourhoods of the endpoints of $I_m$ i.e. when $x \in ([0, \delta_m] \cup (l_m - \delta_m, l_m])$, where $\delta_m$ is sufficiently small,
(iii) $\min\{1, \frac{l_{m+1}}{l_m}\} - (1 - \frac{l_{m+1}}{l_m})^2 \leq f'_m(x) \leq \max\{1, \frac{l_{m+1}}{l_m}\} + (1 - \frac{l_{m+1}}{l_m})^2$
(iv) $f_m$ is the linear function when $x \in (2\delta_m, l_m - 2\delta_m)$ and $f_m(\frac{l_m}{2}) = \frac{l_{m+1}}{2}$.

Define $f : J \to J$ by

$$f(x) = \begin{cases} f_m(x) & \text{when } x \in I_m, \\ \{x + \alpha\} & \text{when } x \in I. \end{cases}$$

It is easy to see that $f$ generates a homeomorphism, denoted by $f$ again, of the circle obtained by the identification of the end points of $J$. $f$ is differentiable, $f'$ is positive and continuous. Hence, $f$ becomes a $C^1$-diffeomorphism of the circle $S^1$.

Now, consider the rectangle $R = J \times J$ and define $g : R \to R$ by

$$g(x, y) = (f(x), f(y)).$$
Note that $g$ maps the rectangle $R_m = I_m \times I_m$ onto the rectangle $R_{m+1} = I_{m+1} \times I_{m+1}$. Let $\{D_m\}_{m \in \mathbb{N}}$ be the sequence of discs centered at $O_m = (\frac{lm}{2}, \frac{lm}{2}) \in R_m$, of radius $4r_m < \frac{lm-26m}{2}$ such that $g(D_m) = D_{m+1}$.

Next, choose $\beta_m \in [0, 2\pi)$ and define $g_m: R \rightarrow R$ likewise in the previous example, i.e. $g_m$ is the rotation of angle $\beta_m$ about $O_m$ on the circle $C_{2r_m}$ centered at $O_m$ of radius $2r_m$ and $g_m(x, y) = (x, y)$ when $(x, y) \in R \setminus D_m$. Obviously if $\beta_m = 0$ then $g_m$ is the identity on $R$.

By the identification of opposite sides of $R$ we obtain the torus $T^2 = S^1 \times S^1$. $g$ and $g_m$ generate homeomorphisms, denoted by $g$ and $g_m$ again, $T^2$. Example 3.2 and this construction show that $g$ and $g_m$, $m \in \mathbb{N}$, are $C^1$-diffeomorphisms of the torus $T^2$.

Next, choose a sequence $\{\beta_m\}_{m \in \mathbb{N}}$ such that for each $m \in \mathbb{N}$, $\beta_m \in [0, 2\pi)$ and $\lim_{m \to \infty} \beta_m = 0$. Let $\{g_m\}_{m \in \mathbb{N}}$ be the associated sequence of diffeomorphisms of the torus $T^2$ (for each $m \in \mathbb{N} g_m$ is defined as above). Define a sequence of diffeomorphisms $\{\varphi_m\}_{m \in \mathbb{N}}$ of the torus $T^2$ by

$$\varphi_0 = g, \quad \varphi_1 = g_1 \circ g, \quad \varphi_2 = g_2 \circ g_1 \circ g, \ldots, \quad \varphi_m = g_m \circ \ldots \circ g_1 \circ g, \ldots$$

There exists $x_0 \in T^2$ (e.g. $x_0 = O_0 \in D_0$) such that $\varphi_0(x_0) = \varphi_1(x_0) = \ldots = \varphi_m(x_0) = \ldots$. So, the sequence $\{\varphi_m(x_0)\}_{m \in \mathbb{N}}$ is convergent. Note that for $x \in D_m$, $(m = 1, 2, \ldots)$

$$\varphi'_n(x) = \begin{cases} (g_m \circ g)'(x) & \text{when } n \geq m, \\ g'(x) & \text{when } n < m. \end{cases}$$

Moreover, from Example 3.2 and the definition of the sequence $\{\beta_m\}$, $g_m$ converges with $m \to \infty$ to the identity, therefore $(g_m \circ g)'(x)$ converges with $m \to \infty$ to $g'(x)$ for all $x \in T^2$. Hence, we have the uniform convergence of the sequence $\{\varphi'_m\}_{m \in \mathbb{N}}$. Consequently, the sequence $\{\varphi_m\}_{m \in \mathbb{N}}$ is uniformly convergent on $T^2$.

Finally, we can define $\varphi: T^2 \to T^2$ by

$$\varphi = \lim_{m \to \infty} \varphi_m$$

It is clear that $\varphi$ is well defined and that $\varphi$ is a $C^1$-diffeomorphism of the torus $T^2$.

Note that for any $m \in \mathbb{N}$, $\varphi$ maps $D_m$ onto $D_{m+1}$ and $C_{2r_m}$ onto $C_{2r_m+1}$, on which it is the rotation of angle $\beta_{m+1}$ about $O_{m+1} \in D_{m+1}$ ($C_{2r_m+1} \subset D_{m+1}$ is the circle centered at $O_{m+1}$ of radius $2r_{m+1}$).

**Theorem 3.4.** If a growth type $\xi$, $[0] < \xi \leq [\exp]$, has a derived growth type $\eta$, then $\xi$ can be realized as a growth type of an orbit $G(x)$ i.e. $\xi = \text{gr}(G(x))$, where $G$ is a suitable finitely generated group of diffeomorphisms of the torus $T^2$ and $x \in T^2$.

**Proof:** From assumption and lemmas in Section 2 we have that $\eta$ is nice and there exist integer-valued $h \in \eta$ and $q \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ we can choose $q_n \in \{1, 2, \ldots, q\}$ such that $h(n+1) = q_nh(n)$. We
can assume that \( h(1) = 1 \). Then \( h(n+1) = q_1q_2 \cdots q_n \). Now, choose sequences
\[
\{\beta_n^1\}_{n \in \mathbb{N}}, \{\beta_n^2\}_{n \in \mathbb{N}}, \ldots, \{\beta_n^q\}_{n \in \mathbb{N}}
\]
where \( \beta_n^i \in [0,2\pi) \) and
\[
\begin{cases}
\beta_n^i = 0 & \text{when } q_n = 1, \\
\beta_n^i = \frac{i-1}{q_1q_2 \cdots q_n} 2\pi & \text{when } q_n \geq i, \\
\beta_n^i = 0 & \text{when } q_n < i.
\end{cases}
\]
for all \( n \in \mathbb{N} \) and \( i = 1, \ldots, q \). Note that \( \lim_{n \to \infty} \beta_n^i = 0 \) for each \( i = 1, 2, \ldots, q \).

For each sequence \( \{\beta_n^i\}_{n \in \mathbb{N}} \) define the associated diffeomorphism \( \varphi^i \) of the torus \( T^2 \) (analogously as \( \varphi \) in Example 3.3). Then, for each \( i \in \{1,2, \ldots, q\} \), \( \varphi^i \) maps \( C_{2r_n} \) onto \( C_{2r_{n+1}} \), on which it is the rotation of angle \( \beta_n^i \), where \( C_{2r_n} \) is the circle centered at \( O_n \in D_n \subset T^2 \) of radius \( 2r_n \) for \( n \in \mathbb{N} \). Set \( G_1 = \{\text{id}_{T^2}, \varphi^1, (\varphi^1)^{-1}, \ldots, \varphi^q, (\varphi^q)^{-1}\} \).

\( G_1 \) is symmetric and generates a group \( G \) of \( C^1 \)-diffeomorphisms of \( T^2 \). Fix a point \( x \in C_{2r_0} \subset D_0 \) and define \( \tilde{h} : \mathbb{N} \to \mathbb{N} \)
\[
\tilde{h}(1) = \#G_1(x) - 1,
\tilde{h}(n+1) = \#G_{n+1}(x) - \#G_n(x).
\]
Put
\[
G_1^+ = \{\text{id}_{T^2}, \varphi^1, \ldots, \varphi^q\}, \quad G_1^- = \{\text{id}_{T^2}, (\varphi^1)^{-1}, \ldots, (\varphi^q)^{-1}\}.
\]
Then for all \( n \in \mathbb{N} \) we have
\[
\#G_{n+1}(x) - \#G_n(x) = \#G_1(G_n(x)) - \#G_n(x)
= \#G_1^+(G_n(x)) - \#G_n(x)
+ \#G_1^-(G_n(x)) - \#G_n(x).
\]
By the definition of sequences \( \{\beta_n^i\} \) and the construction of diffeomorphisms \( \varphi^i (i = 1, \ldots, q) \) we can see that
\[
\#G_1^+(G_n(x)) - \#G_n(x) = h(n+2),
\#G_1^-(G_n(x)) - \#G_n(x) = (h(n+1) - h(n)) + (h(n) - h(n-1))
+ \ldots + (h(2) - h(1)) + h(1)
= h(n+1).
\]
Hence
\[
\#G_{n+1}(x) - \#G_n(x) = h(n+2) + h(n+1).
\]
Moreover,
\[
\#G_1(x) - 1 = \#G_1^+(x) - 1 + \#G_1^-(x) - 1 = h(2) + h(1).
\]
So, by the definition of the function \( \tilde{h} \), we have
\[
\tilde{h}(1) = h(2) + h(1),
\tilde{h}(n+1) = h(n+2) + h(n+1) \text{ for all } n \in \mathbb{N}.
\]
We obtain that $\tilde{h} \in I$ and $[\tilde{h}] = [h]$. Hence $\tilde{h} \in \eta$ and $[\Sigma \tilde{h}] = \xi$. Since $\# G_n(x) = \Sigma \tilde{h}(n) + 1$, then $\text{gr}(G(x)) = [\Sigma \tilde{h}]$. Consequently $\xi = \text{gr}(G(x))$. □

**Corollary 3.5.** If a growth type $\xi$, $[0] < \xi \leq [\exp]$, has a derived growth type $\eta$ then there exists a $C^1$-foliation $\mathcal{F}$ of a suitable compact Riemannian manifold $M$ containing a leaf $L$ which has the growth type $\text{gr}(L) = \xi$ i.e. $\xi$ can be realized as the growth type of a leaf.

**Proof:** From the previous theorem we have that $\xi$ can be realized as the growth type of an orbit of a finitely generated group $G$ of diffeomorphisms of the torus $T^2$ i.e. $\xi = \text{gr}(G(x))$ for certain $x \in T^2$. Let $G_1 = \{\text{id}_{T^2}, \varphi_1, \varphi_1^{-1}, \ldots, \varphi_q, \varphi_q^{-1}\}$ be the symmetric generating set and let $B = \Sigma_q$ be a compact oriented surface of genus $q$. Then

$$\pi_1(B) = \langle a_1, b_1, \ldots, a_q, b_q : a_1b_1a_1^{-1}b_1^{-1}\ldots a_qb_qa_q^{-1}b_q^{-1}\rangle$$

Put $\Gamma = \pi_1(B)$ and take homomorphism $h: \Gamma \to Diff^1(T^2)$ such that

$$h(a_i) = \varphi_i, \quad h(b_i) = \text{id}_{T^2} \quad (i = 1, \ldots, q).$$

Then $h(\Gamma) = G$ and by the suspension of the homomorphism $h$ we obtain a foliated manifold $(M, \mathcal{F})$ (see e.g. [CdC], [Wa]). $\mathcal{F}$ is a $C^1$-foliation of codimension 2. The holonomy pseudogroup $\mathcal{H}$ of $\mathcal{F}$ is isomorphic to a pseudogroup $G(G)$ (generated by group $G$). Hence $\text{gr}(\mathcal{H}(x)) = \text{gr}(G(x))$, where $x \in T^2$. If $L \in \mathcal{F}$ is a leaf passing through $x$ then, from Lemma 2.6, $\text{gr}(L) = \text{gr}(\mathcal{H}(x))$. Consequently $\xi = \text{gr}(L)$ and $\xi$ is realized as the growth type of a leaf. □

**Remark 3.6.** In the above proof we have obtained a foliation of codimension 2. Note that if we take a compact $n$-manifold $N$ and foliated manifold $(M, \mathcal{F})$ obtained in the above proof then we can obtain a foliation of the manifold $M \times N$ of codimension $n + 2$ with leaves $L \times \{y\}$, where $y \in N$. Then $\text{gr}(L) = \text{gr}(L \times \{y\})$.

**References**


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